Average length of the longest k-alternating subsequence

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Abstract. We prove a conjecture of Drew Armstrong on the average maximal length of k-alternating subsequence of permutations. The $k=1$ case is a well-known result of Richard Stanley.

1. Introduction

We fix positive integers n, k with $n \geq 2$ and $1 \leq k \leq n-1$.

Let $w = w_1 w_2 \cdots w_n$ in $\mathfrak{S}_n$, the permutation group of $[1, n]$. A subsequence $w_{i_1} \cdots w_{i_s}$ of w is *alternating* if $w_{i_1} > w_{i_2} < w_{i_3} \cdots$. We call it *$k$-alternating* if moreover each neighboring pair satisfies $|w_{i_j} - w_{i_{j+1}}| \geq k$. We call the maximal length (which is the number of elements) of the k-alternating subsequences of w the *k-alternating length* of w and denote it as $as_k(w)$ [**1**]. We denote the average of the k-alternating length of permutations in $\mathfrak{S}_n$ by $E_n(as_k)$; i.e., $E_n(as_k) = \frac{1}{n!}\sum_{w \in \mathfrak{S}_n} as_k(w)$. We prove the following result which was conjectured by Drew Armstrong [**1**]:

Theorem 1.1. *For integers n, k with $n \geq k+1 \geq 2$, the average k-alternating length of permutations in $\mathfrak{S}_n$ is*

$$E_n(as_k) = \frac{4(n-k)+5}{6}. \tag{1.1}$$

The special case when $k=1$ is a result of Stanley [**3, 4**]. Igor Pak and Robin Pemantle proved that $E_n(as_k)$ is asymptotically $2(n-k)/3$ using a probabilistic method [**2**].

We call a subsequence satisfying $w_{i_1} < w_{i_2} > w_{i_3} \cdots$ *reverse alternating*. We say a subsequence is *zigzagging* if it is either alternating or reverse alternating. Then we similarly define a *k-zigzagging subsequence* and the *k-zigzagging length* $zs_k(w)$. We denote the average k-zigzagging length of permutations in $\mathfrak{S}_n$ by $E_n(zs_k)$.

Note that the *swapping map* $I : w_1 w_2 \cdots w_n \to (n+1-w_1)(n+1-w_2) \cdots (n+1-w_n)$ is an involution interchanging alternating subsequences

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and reverse alternating subsequences. Thus exactly half of the permutations $w \in \mathfrak{S}_n$ have k-zigzagging length that is one more than their k-alternating length, while for the other half the two lengths are equal. Therefore $E_n(zs_k) = E_n(as_k) + 1/2$. Hence we have:

LEMMA 1.2. *The formula* (1.1) *is equivalent to the formula*

$$E_n(zs_k) = \frac{2(n-k)+4}{3}. \tag{1.2}$$

Let us take a look at the $k = 1$ case of the proof to get some ideas about our proof. In this case, the zigzagging length of w is equal to the number of its peaks and valleys, where w_i is a peak (respectively a valley) if it is greater (respectively less) than its one or two neighbors. We see that w_1 and w_n each is a peak or a valley. With a little thought, one sees that the probability that w_i is a peak or a valley is $2/3$ when $1 < i < n$. Now we see that $E_n(zs_1) = 1 + (n-2) \times \frac{2}{3} + 1 = \frac{2n+2}{3}$, in agreement with (1.2). (The author learned this proof from Richard Stanley, who learned it from Miklos Bóna. See Section 4 of [**3**].)

Our proof is similar to this argument. We first define the k-peaks and k-valleys of a permutation, which are the original peaks and valleys when $k = 1$. We prove that the k-zigzagging length of a permutation is equal to the number of its k-peaks and k-valleys. Then we count the probability that a number j is a k-peak in a permutation. Finally, we prove formula (1.2) which is equivalent to (1.1).

2. k-peaks and k-valleys

DEFINITION 2.1. Let $w = w_1w_2 \cdots w_n \in \mathfrak{S}_n$ and $n > k \geq 1$. We call a section $w_sw_{s+1} \cdots w_t$ in w a *k-up* (respectively a *k-down*) if $s < t$ and $w_t - w_s \geq k$ (respectively $w_s - w_t \geq k$). We say a section $w_iw_{i+1} \cdots w_j$ $(i < j)$ of w is *k-ascending* if it satisfies the following:

[1] $w_i = \min\{w_i, w_{i+1}, \ldots, w_j\}$, $w_j = \max\{w_i, w_{i+1}, \ldots, w_j\}$;
[2] $w_j - w_i \geq k$; i.e., $w_i \cdots w_j$ is a k-up;
[3] if $i \leq s < t \leq j$ then $w_s - w_t < k$; i.e., there is no k-down in $w_i \cdots w_j$.

If moreover $w_i \cdots w_j$ is not contained in another k-ascending section, we call it a *maximal k-ascending* section. In this case, we call w_i a *k-valley* of w and w_j a *k-peak* of w.

Similarly, we define k-down, k-descending, and maximal k-descending. For a maximal k-descending section $w_i \cdots w_j$ of w we also call w_i a k-peak of w and w_j a k-valley of w.

EXAMPLE 2.2. Let $w = w_1w_2 \cdots w_n \in \mathfrak{S}_n$. We see that if $1 \leq j \leq k$, then the number j is not a k-peak in w.

EXAMPLE 2.3. Consider the permutation $w = 214386759 \in \mathfrak{S}_9$. We see that the number 2 is not in a maximal 3-ascending section or a maximal

3-descending section. The sections 1438 and 59 are maximal 3-ascending sections, while 8675 is a maximal 3-descending section. Finally, 1859 is a longest 3-zigzagging subsequence of w.

This example suggests that a permutation can be viewed as a chain of alternating maximal k-ascending sections and maximal k-descending sections. The link points are those k-valleys and k-peaks. It is possible, however, that a beginning section or a ending section is not covered by this chain. Most importantly, we also see that the subsequence formed by the k-peaks and k-valleys is a longest k-zigzagging subsequence of w (see Proposition 2.8). We will only need to count the total number of the k-peaks, because the total number of k-peaks of all permutations is equal to that of the k-valleys, which can be seen applying the swapping map I.

We have the following properties to prolong a k-ascending section. Using the swapping map I, one finds similar properties for a k-descending section.

LEMMA 2.4. *Let a section $w_i \cdots w_j$ in $w = w_1 \cdots w_n$ be k-ascending.*

(1) *If there is a $t > j$ with $w_j < w_t$ and no k-down in $w_j \cdots w_t$ then the k-ascending section $w_i \cdots w_j$ can be prolonged from the right, i.e., there is a $j < t' \leq t$ such that $w_i \cdots w_j \cdots w_{t'}$ is k-ascending;*
(2) *If there is a $s < i$ with $w_s < w_i$ and no k-down in $w_s \cdots w_i$ then the k-ascending section $w_i \cdots w_j$ can be prolonged from the left, i.e.,there is an $s \leq s' < i$ such that $w_{s'} \cdots w_i \cdots w_j$ is k-ascending.*

PROOF. For the first statement, take $w_{t'} = \max\{w_j, w_{j+1}, \ldots, w_t\}$. It is easy to verify that $w_i \cdots w_j \cdots w_{t'}$ is a desired k-ascending section. The second statement is completely analogous. $\square$

The following property says that a k-up contains a k-ascending section. There is a similar fact for a k-down.

LEMMA 2.5. *Let (w_i, w_j) be a k-up. Let $i \leq i' < j' \leq j$ such that $w_{i'} \cdots w_{j'}$ is a shortest (i.e., $|i' - j'|$ is minimal) k-up. Then $w_{i'} \cdots w_{j'}$ is a k-ascending section.*

PROOF. This can easily be verified by definition. $\square$

LEMMA 2.6. *The intersection of a maximal k-ascending section and a maximal k-descending section is empty or a one-element set. Two distinct maximal k-ascending sections do not intersect.*

PROOF. The first statement is easy by considering the maximum and minimum of the two sections.

The second statement follows from Lemma 2.4. $\square$

The following result together with Lemma 2.5 tells us that every permutation w is covered by its maximal k-ascending sections and maximal k-descending sections, except possibly a beginning section and/or an ending section of w.

Lemma 2.7. *Let $\gamma = w_i w_{i+1} \cdots w_j$ and $\delta = w_{i'} w_{i'+1} \cdots w_{j'}$ each be a maximal k-ascending section or a maximal k-descending section. If $j < i'$ then there is a k-up or a k-down in $w_j w_{j+1} \cdots w_{i'}$.*

Proof. If there is no k-up or k-down in $w_j \cdots w_{i'}$, Lemma 2.4 will always allow us to prolong one of the two sections γ and δ, a contradiction to the maximality of γ and δ.

For example, let us consider the case that both γ and δ are maximal k-ascending (and there is no k-down or k-up in $w_j \cdots w_{i'}$). Then $w_i < w_{i'}$. (Otherwise, $w_j \cdots w_{i'}$ is already a k-down as $w_j - w_{i'} > w_j - w_i \geq k$.) Moreover, there is no k-down in $w_i \cdots w_{i'}$. Thus $w_{i'} \cdots w_{j'}$ can be prolonged from the left by Lemma 2.4. □

Proposition 2.8. *The subsequence of a permutation formed by the k-peaks and k-valleys is a longest k-zigzagging subsequence. Thus the average k-zigzagging length of permutations is two times the average number of k-peaks of permutations.*

Proof. Let $w_{i_1} w_{i_2} \cdots w_{i_s}$ be the subsequence formed by the k-peaks and k-valleys of w. Let $\gamma_r = w_{i_r} \cdots w_{i_{r+1}}$ $(r = 1, 2, \ldots, s-1)$. We see that w is a union of these $s+1$ sections $\gamma_0, \gamma_1, \ldots, \gamma_{s-1}, \gamma_s$, where $\gamma_1, \cdots, \gamma_{s-1}$ is an alternating sequence of maximal k-ascending sections and maximal k-descending sections. (The (beginning) section of w, $\gamma_0 = w_1 \cdots w_{i_1}$, is a single element if $i_1 = 1$. The (ending) section of w, $\gamma_s = w_{i_s} \cdots w_n$, is a single element if $i_s = n$.) To form a k-zigzagging subsequence of w, one can take at most one element from each of γ_0 and γ_s. One can take at most two elements from each of $\gamma_1, \cdots, \gamma_{s-1}$; but to take two elements from each of γ_t, γ_{t+1}, one has to take the link point $w_{i_{t+1}}$. Thus we see that taking the k-peaks and k-valleys is one way to have the maximum length of k-zigzagging subsequence.

The second statement now follows because the total number of k-peaks of all permutations is equal to that of k-valleys. □

3. A characterization of k-peaks and the proof of the theorem

We will need the following characterization of k-peaks.

Proposition 3.1. *Let $w = w_1 \cdots w_n \in \mathfrak{S}_n$, $i \in [1, n]$ and $1 \leq k \leq n-1$. Then w_i is a k-peak if and only if it satisfies the following two properties.*

1. *If there is an $s > i$ with $w_s > w_i$, then there is a k-down $w_i \cdots w_j$ in $w_i \cdots w_s$.*
2. *If there is an $s < i$ with $w_s > w_i$, then there is a k-up $w_j \cdots w_i$ in $w_s \cdots w_i$.*

Remark 3.2. (1) Note that if $w_i = n$ than it satisfies these two properties for all positive integers k. Therefore the number n appears as a k-peak for all $1 \leq k \leq n-1$. (2) By this proposition, a k-peak is also a k'-peak if $1 \leq k' \leq k \leq n-1$.

PROOF OF PROPOSITION 3.1. Proof of "only if": Let w_i be a k-peak. Then it is the ending of a maximal k-ascending section and/or the beginning of a k-descending section. Let us consider the case that it is the ending of a maximal k-ascending section $w_{i'} \cdots w_i$; the other case can be done similarly.

First w_i satisfies the second property. Now assume that it does not satisfy the first property. Then we can take the minimum s such that $s > i$, $w_s > w_i$ and there is no k-down $w_i \cdots w_j$ in $w_i \cdots w_s$. Then $w_i > w_{s'}$ for $i < s' < s$ by the minimality of s. Therefore there is no k-down in $w_i \cdots w_s$. (Because if $w_{j'} \cdots w_j$ is a k-down in $w_i \cdots w_s$, then so is $w_i \cdots w_j$ as $w_i > w_{j'}$). By Lemma 2.4 we can prolong the maximal k-ascending section $w_{i'} \cdots w_i$ from the right, a contradiction.

Proof of "if": First there is at least one k-down $w_i \cdots w_j$ or one k-up $w_j \cdots w_i$ (no matter whether w_i equals n or not). Let us prove the case when there is a k-up $w_j \cdots w_i$; the other case is proved similarly. Let w_t be the closest element to w_i (so $|i-t|$ is minimum) such that $w_t \cdots w_i$ is a k-up. We show in the following that $w_t \cdots w_i$ is k-ascending.

First, w_t is the minimum in $\{w_t, \ldots, w_i\}$ by the choice of it. Also w_i is the maximum in $\{w_t, \ldots, w_i\}$. Otherwise, let w_s in $w_t \cdots w_i$ be greater than w_i; thus there is a k-up $w_{s'} \cdots w_i$ in $w_s \cdots w_i$. This $w_{s'}$ is closer to w_i than w_t is, contradicting to the choice of w_t. Second, $w_t \cdots w_i$ is known to be a k-up. Third, there is no k-down in $w_t \cdots w_i$. Otherwise, let $w_r \cdots w_s$ be a k-down in $w_t \cdots w_i$. Then $w_i - w_s > w_r - w_s \geq k$ and thus $w_s \cdots w_i$ is a k-up and w_s is closer to w_i than w_t is, a contradiction.

Now as $w_t \cdots w_i$ is a k-ascending section; it is thus contained in a maximal k-ascending section $w_{t'} \cdots w_{i'}$. If $i' > i$, then $w_{i'} > w_i$, and thus there is a k-down $w_i \cdots w_r$ in $w_i \cdots w_{i'}$ (by the first property), which contradicts the fact that $w_{t'} \cdots w_{i'}$ is a (maximal) k-ascending section. Therefore $i' = i$ and hence w_i is a k-peak, as desired. □

Now we apply Proposition 3.1 to find the probability that a number j appears as a k-peak in a permutation in $\mathfrak{S}_n$. For instance, by this proposition, we know that the probability of n being a k-peak is 1.

PROPOSITION 3.3. *Let $1 \leq j \leq n$ and $1 \leq k \leq n-1$. Let $p_{n,k}(j)$ be the probability that j is a k-peak of a randomly selected permutation in $\mathfrak{S}_n$. We have*

$$p_{n,k}(j) = \begin{cases} 0 & \text{if } j \leq k \\ \frac{(j-k)(j-k+1)}{(n-k)(n-k-1)} & \text{if } j > k. \end{cases}$$

PROOF. The case $j \leq k$ is known by Example 2.2 or by Proposition 3.1.

Let us consider the case $j > k$. We partition the set $[1,n] - \{j\}$ into three subsets:

$$\begin{aligned} A &= \{l : 1 \leq l \leq j-k\} \\ B &= \{l : j-k+1 \leq l \leq j-1\} \\ C &= \{l : j+1 \leq l \leq n\}. \end{aligned}$$

To form a permutation, let us first arrange $A \cup \{j\}$ on a row $a_1 a_2 \cdots a_{j-k+1}$, then we insert the elements from the set $B \cup C$ one by one into this row. We first insert the number $j+1$ into $a_1 a_2 \cdots a_{j-k+1}$. There are $j-k+2$ positions to put: put it to the left of a_1, put it between a_1 and a_2, put it between a_2 and a_3, on and on, and put it to the right of a_{j-k+1}. We form a new row with $j+k+2$ elements. Then we put the number $j+2$ into this new row, and there are $j-k+3$ positions to do this. Keep doing this until we exhaust all elements in C; then do elements from B.

We see that all permutations can be obtained this way. But to make j a k-peak, it is sufficient and necessary that we do not put any element from C next to j. This is because Proposition 3.1 tells us that between j and an element from C there should be at least an element from A. The insertion of elements from B will not change the property that j is a k-peak or not.

Therefore when first adding $j+1$, there are $j-k$ *right* positions out of the $j-k+2$ positions to put it. When adding $j+2$, there are $j-k+1$ *right* ways out of the $j-k+3$ ways to do so. So on and so forth, until when adding n, there are $n-k-1$ *right* ways out of the $n-k+1$ ways to do so. Therefore the probability of j being a k-peak is as follows:

$$\begin{aligned} p_{n,k}(j) &= \frac{j-k}{j-k+2} \times \frac{j-k+1}{j-k+3} \times \cdots \times \frac{n-k-1}{n-k+1} \\ &= \frac{(j-k)(j-k+1)}{(n-k)(n-k+1)}. \end{aligned}$$

□

Proof of Theorem 1.1. As the probability of j being a k-peak in a permutation $w \in \mathfrak{S}_n$ is $p_{n,k}(j)$, the average number of k-peaks of a permutations in $\mathfrak{S}_n$ is $\sum_{j=1}^{n} p_{n,k}(j)$. By Propositions 2.8 and 3.3, we have

$$\begin{aligned} E_n(zs_k) &= 2\sum_{j=1}^{n} p_{n,k}(j) \\ &= 2\sum_{j=k+1}^{n} \frac{(j-k)(j-k+1)}{(n-k)(n-k+1)} \\ &= \frac{2(n-k)+4}{3}. \end{aligned}$$

This is formula (1.2), which is equivalent to (1.1) by Lemma 1.2. □

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